

Cubic Spline Questions

5. Find natural cubic splines which interpolate the following dataset of (x,y) points: (3.0,2.5), (4.5,1.0), (7.0,2.5), (9.0,0.5); estimate the value for x=5.

Solution:

Step 1: Use the n-2 cubic spline equations to find the second derivatives y'' (slide 80):

$$\frac{x_j - x_{j-1}}{6} y''_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} y''_j + \frac{x_{j+1} - x_j}{6} y''_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

$$j = 2: \quad \frac{4.5 - 3.0}{6} y''_1 + \frac{7.0 - 3.0}{3} y''_2 + \frac{7.0 - 4.5}{6} y''_3 = \frac{2.5 - 1.0}{7.0 - 4.5} - \frac{1.0 - 2.5}{4.5 - 3.0}$$

$$\frac{1}{4} y''_1 + \frac{4}{3} y''_2 + \frac{5}{12} y''_3 = \frac{8}{5}$$

$$j = 3: \quad \frac{7.0 - 4.5}{6} y''_2 + \frac{9.0 - 4.5}{3} y''_3 + \frac{9.0 - 7.0}{6} y''_4 = \frac{0.5 - 2.5}{9.0 - 7.0} - \frac{2.5 - 1.0}{7.0 - 4.5}$$

$$\frac{5}{12} y''_2 + \frac{3}{2} y''_3 + \frac{1}{3} y''_4 = -\frac{8}{5}$$

Natural splines have $y''_1 = y''_4 = 0$ so the system simplifies to:

$$\begin{array}{l} y''_1 = 0 \\ \frac{4}{3} y''_2 + \frac{5}{12} y''_3 = \frac{8}{5} \\ \frac{5}{12} y''_2 + \frac{3}{2} y''_3 = -\frac{8}{5} \\ y''_4 = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} y''_1 = 0 \\ y''_2 = 1.6791 \\ y''_3 = -1.5331 \\ y''_4 = 0 \end{array}$$

Step 2: Apply these solutions to obtain the cubic spline equations from formulae for A, B, C, D (slides 75 & 77) for the three subintervals (i.e., with $j = 1, 2, 3$). For $j = 1$:

$$A = \frac{x_2 - x}{x_2 - x_1} = \frac{4.5 - x}{4.5 - 3.0} = 3 - \frac{2}{3}x$$

$$B = 1 - A = 1 - (3 - \frac{2}{3}x) = \frac{2}{3}x - 2$$

$$C = \frac{1}{6}(A^3 - A)(x_2 - x_1)^2 = \frac{1}{6}[(3 - \frac{2}{3}x)^3 - (3 - \frac{2}{3}x)](4.5 - 3)^2$$

$$D = \frac{1}{6}(B^3 - B)(x_2 - x_1)^2 = \frac{1}{6}[(\frac{2}{3}x - 2)^3 - (\frac{2}{3}x - 2)](4.5 - 3)^2$$

The spline equation, which applied for the first subinterval $3.0 < x < 4.5$, is then given by:

$$\begin{aligned}
 y &= Ay_1 + By_2 + Cy_1'' + Dy_2'' \\
 &= (3 - \frac{2}{3}x)(2.5) + (\frac{2}{3}x - 2)(1.0) + 0 + \frac{2.25}{6} [(\frac{2}{3}x - 2)^3 - (\frac{2}{3}x - 2)](1.6791) \\
 &= 0.1866(x - 3)^3 + 0.2469(x - 3) - 1.6667(x - 4.5)
 \end{aligned}
 \tag{S1}$$

For $j = 2$:

$$\begin{aligned}
 A &= \frac{x_3 - x}{x_3 - x_2} = \frac{7.0 - x}{7.0 - 4.5} = \frac{14}{5} - \frac{2}{5}x \\
 B &= 1 - A = 1 - (\frac{14}{5} - \frac{2}{5}x) = \frac{2}{5}x - \frac{9}{5} \\
 C &= \frac{1}{6}(A^3 - A)(x_3 - x_2)^2 = \frac{1}{6}[(\frac{14}{5} - \frac{2}{5}x)^3 - (\frac{14}{5} - \frac{2}{5}x)](7.0 - 4.5)^2 \\
 D &= \frac{1}{6}(B^3 - B)(x_3 - x_2)^2 = \frac{1}{6}[(\frac{2}{5}x - \frac{9}{5})^3 - (\frac{2}{5}x - \frac{9}{5})](7.0 - 4.5)^2
 \end{aligned}$$

The spline equation, which applied for the second subinterval $4.5 < x < 7.0$, is then given by:

$$\begin{aligned}
 y &= Ay_2 + By_3 + Cy_2'' + Dy_3'' \\
 &= (\frac{14}{5} - \frac{2}{5}x)(1.0) + (\frac{2}{5}x - \frac{9}{5})(2.5) + \frac{6.25}{6} [(\frac{14}{5} - \frac{2}{5}x)^3 - (\frac{14}{5} - \frac{2}{5}x)](1.6791) \\
 &\quad + \frac{6.25}{6} [(\frac{2}{5}x - \frac{9}{5})^3 - (\frac{2}{5}x - \frac{9}{5})](-1.5331) \\
 &= -0.1022(x - 4.5)^3 - 0.1119(x - 7)^3 + 1.6388(x - 4.5) + 0.2996(x - 7)
 \end{aligned}
 \tag{S2}$$

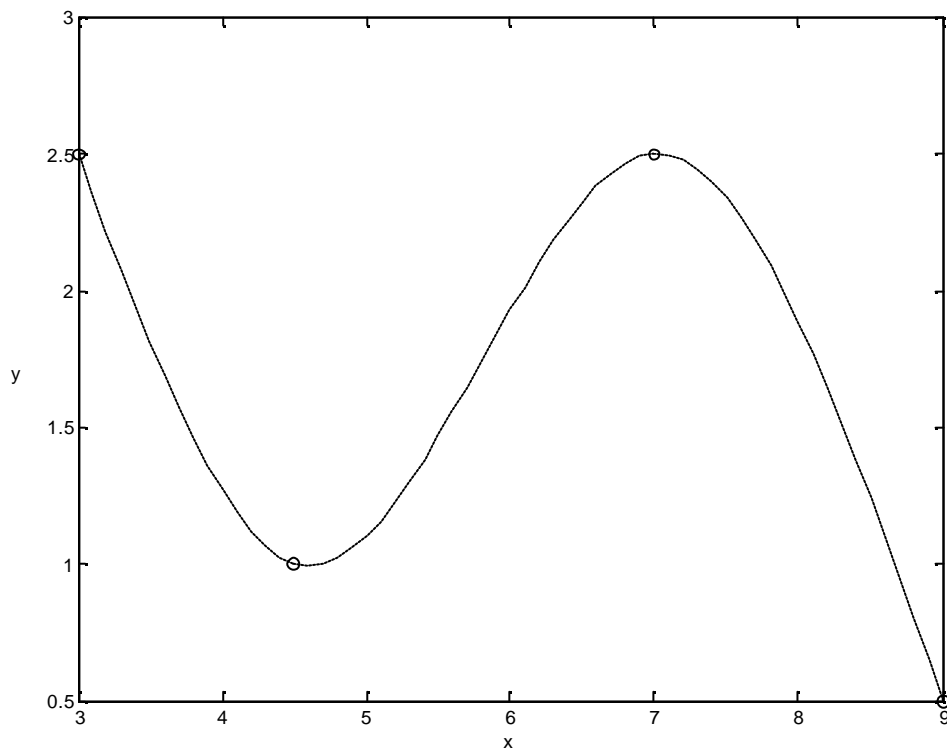
For $j = 3$:

$$\begin{aligned}
 A &= \frac{x_4 - x}{x_4 - x_3} = \frac{9.0 - x}{9.0 - 7.0} = \frac{9}{2} - \frac{1}{2}x \\
 B &= 1 - A = 1 - (\frac{9}{2} - \frac{1}{2}x) = \frac{1}{2}x - \frac{7}{2} \\
 C &= \frac{1}{6}(A^3 - A)(x_4 - x_3)^2 = \frac{1}{6}[(\frac{9}{2} - \frac{1}{2}x)^3 - (\frac{9}{2} - \frac{1}{2}x)](9.0 - 7.0)^2 \\
 D &= \frac{1}{6}(B^3 - B)(x_4 - x_3)^2 = \frac{1}{6}[(\frac{1}{2}x - \frac{7}{2})^3 - (\frac{1}{2}x - \frac{7}{2})](9.0 - 7.0)^2
 \end{aligned}$$

The spline equation, which applied for the third subinterval $7.0 < x < 9.0$, is then given by:

$$\begin{aligned}
 y &= Ay_3 + By_4 + Cy_3'' + Dy_4'' \\
 &= \left(\frac{9}{2} - \frac{1}{2}x\right)(2.5) + \left(\frac{1}{2}x - \frac{7}{2}\right)(0.5) + \frac{2}{3}\left[\left(\frac{9}{2} - \frac{1}{2}x\right)^3 - \left(\frac{9}{2} - \frac{1}{2}x\right)\right](-1.5331) + 0 \\
 &= 0.1278(x-9)^3 + 0.25(x-7) - 1.7610(x-9)
 \end{aligned}
 \tag{S3}$$

The three equations S1, S2 and S3 constitute the natural cubic spline equations for y over the interval $3.0 < x < 9.0$...



To estimate the y value at $x = 5$, use the spline equation for the second sub-interval (S2):

$$\begin{aligned}
 y_{x=5} &= -0.1022(x-4.5)^3 - 0.1119(x-7)^3 + 1.6388(x-4.5) + 0.2996(x-7) \\
 &= -0.1022(5-4.5)^3 - 0.1119(5-7)^3 + 1.6388(5-4.5) + 0.2996(5-7) \\
 &= 1.1026
 \end{aligned}$$

6. Find the interpolating cubic splines for the five logarithmic breakpoints (1, ln 1), (2, ln 2), (3, ln 3), (4, ln 4), and (6, ln 6). Use [i.e. three exercises to do here]: (a) natural endpoint conditions; (b) specified end slopes of 1 at x=1 and 1/6 at x=6; and (c) not-a-knot conditions. Compare the graphs corresponding to the three different endpoint conditions. For whichever of these endpoint conditions can be used with the built-in Matlab function, compare your solution to that found with Matlab.

Solution:

Step 1: Use the n-2 cubic spline equations to find the second derivatives y'' . We end up with the following system of equations (denoted "System 1"):

$$\frac{x_j - x_{j-1}}{6} y''_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} y''_j + \frac{x_{j+1} - x_j}{6} y''_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

$$j = 2: \quad \frac{2-1}{6} y''_1 + \frac{3-1}{3} y''_2 + \frac{3-2}{6} y''_3 = \frac{\ln 3 - \ln 2}{3-2} - \frac{\ln 2 - \ln 1}{2-1}$$

$$\frac{1}{6} y''_1 + \frac{2}{3} y''_2 + \frac{1}{6} y''_3 = \ln \frac{3}{4}$$

$$j = 3: \quad \frac{3-2}{6} y''_2 + \frac{4-2}{3} y''_3 + \frac{4-3}{6} y''_4 = \frac{\ln 4 - \ln 3}{4-3} - \frac{\ln 3 - \ln 2}{3-2}$$

$$\frac{1}{6} y''_2 + \frac{2}{3} y''_3 + \frac{1}{6} y''_4 = \ln \frac{8}{9}$$

$$j = 4: \quad \frac{4-3}{6} y''_3 + \frac{6-3}{3} y''_4 + \frac{6-4}{6} y''_5 = \frac{\ln 6 - \ln 4}{6-4} - \frac{\ln 4 - \ln 3}{4-3}$$

$$\frac{1}{6} y''_3 + y''_4 + \frac{1}{3} y''_5 = \ln \frac{3\sqrt{6}}{8}$$

Step 2: Find A , B , C , D (slides 75 & 77) for the four subintervals. For $j = 1$:

$$A_1 = \frac{x_2 - x}{x_2 - x_1} = \frac{2 - x}{2 - 1} = 2 - x$$

$$B_1 = 1 - A_1 = 1 - (2 - x) = x - 1$$

$$C_1 = \frac{1}{6} (A_1^3 - A_1) (x_2 - x_1)^2 = \frac{1}{6} [(2 - x)^3 - (2 - x)] (2 - 1)^2$$

$$D_1 = \frac{1}{6} (B_1^3 - B_1) (x_2 - x_1)^2 = \frac{1}{6} [(x - 1)^3 - (x - 1)] (2 - 1)^2$$

For $j = 2$:

$$A_2 = \frac{x_3 - x}{x_3 - x_2} = \frac{3 - x}{3 - 2} = 3 - x$$

$$B_2 = 1 - A_2 = 1 - (3 - x) = x - 2$$

$$C_2 = \frac{1}{6}(A_2^3 - A_2)(x_3 - x_2)^2 = \frac{1}{6}[(3 - x)^3 - (3 - x)](3 - 2)^2$$

$$D_2 = \frac{1}{6}(B_2^3 - B_2)(x_3 - x_2)^2 = \frac{1}{6}[(x - 2)^3 - (x - 2)](3 - 2)^2$$

For $j = 3$:

$$A_3 = \frac{x_4 - x}{x_4 - x_3} = \frac{4 - x}{4 - 3} = 4 - x$$

$$B_3 = 1 - A_3 = 1 - (4 - x) = x - 3$$

$$C_3 = \frac{1}{6}(A_3^3 - A_3)(x_4 - x_3)^2 = \frac{1}{6}[(4 - x)^3 - (4 - x)](4 - 3)^2$$

$$D_3 = \frac{1}{6}(B_3^3 - B_3)(x_4 - x_3)^2 = \frac{1}{6}[(x - 3)^3 - (x - 3)](4 - 3)^2$$

For $j = 4$:

$$A_4 = \frac{x_5 - x}{x_5 - x_4} = \frac{6 - x}{6 - 4} = 3 - \frac{1}{2}x$$

$$B_4 = 1 - A_4 = 1 - (3 - \frac{1}{2}x) = \frac{1}{2}x - 2$$

$$C_4 = \frac{1}{6}(A_4^3 - A_4)(x_5 - x_4)^2 = \frac{1}{6}[(3 - \frac{1}{2}x)^3 - (3 - \frac{1}{2}x)](6 - 4)^2$$

$$D_4 = \frac{1}{6}(B_4^3 - B_4)(x_5 - x_4)^2 = \frac{1}{6}[(\frac{1}{2}x - 2)^3 - (\frac{1}{2}x - 2)](6 - 4)^2$$

(a) Using natural endpoint conditions...

Natural splines have $y_1' = y_5' = 0$ so "System 1" simplifies to:

$$\begin{array}{rcl}
 y_1'' = 0 & & y_1'' = 0 \\
 \frac{2}{3}y_2'' + \frac{1}{6}y_3'' = \ln \frac{3}{4} & & y_2'' = -0.4182 \\
 \frac{1}{6}y_2'' + \frac{2}{3}y_3'' + \frac{1}{6}y_4'' = \ln \frac{8}{9} & \Rightarrow & y_3'' = -0.05310 \\
 \frac{1}{6}y_3'' + y_4'' = \ln \frac{3\sqrt{6}}{8} & & y_4'' = -0.07610 \\
 y_5' = 0 & & y_5'' = 0
 \end{array}$$

The spline equation, which applied for the first subinterval $1 < x < 2$, is then given by:

$$\begin{aligned}
 y &= A_1y_1 + B_1y_2 + C_1y_1'' + D_1y_2'' \\
 &= (2-x)(\ln 1) + (x-1)(\ln 2) + 0 + \frac{1}{6}[(x-1)^3 - (x-1)](-0.4182) \\
 &= -0.0697(x-1)^3 + 0.7628(x-1) \qquad \qquad \qquad \text{S1}
 \end{aligned}$$

The spline equation, which applied for the second subinterval $2 < x < 3$, is then given by:

$$\begin{aligned}
 y &= A_2y_2 + B_2y_3 + C_2y_2'' + D_2y_3'' \\
 &= (3-x)(\ln 2) + (x-2)(\ln 3) + \frac{1}{6}[(3-x)^3 - (3-x)](-0.4182) + \frac{1}{6}[(x-2)^3 - (x-2)](-0.05310) \\
 &= -0.00885(x-2)^3 + 0.0697(x-3)^3 + 1.1075(x-2) - 0.7628(x-3) \qquad \qquad \qquad \text{S2}
 \end{aligned}$$

The spline equation, which applied for the third subinterval $3 < x < 4$, is then given by:

$$\begin{aligned}
 y &= A_3y_3 + B_3y_4 + C_3y_3'' + D_3y_4'' \\
 &= (4-x)(\ln 3) + (x-3)(\ln 4) + \frac{1}{6}[(4-x)^3 - (4-x)](-0.05310) + \frac{1}{6}[(x-3)^3 - (x-3)](-0.07610) \\
 &= -0.01268(x-3)^3 + 0.00885(x-4)^3 + 1.3990(x-3) - 1.1075(x-4) \qquad \qquad \qquad \text{S3}
 \end{aligned}$$

The spline equation, which applied for the fourth subinterval $4 < x < 6$, is then given by:

$$\begin{aligned}
 y &= A_4y_4 + B_4y_5 + C_4y_4'' + D_4y_5'' \\
 &= (3 - \frac{1}{2}x)(\ln 4) + (\frac{1}{2}x - 2)(\ln 6) + \frac{2}{3}[(3 - \frac{1}{2}x)^3 - (3 - \frac{1}{2}x)](-0.07610) + 0 \\
 &= 0.006342(x-6)^3 + 0.8959(x-4) - 0.7185(x-6) \qquad \qquad \qquad \text{S4}
 \end{aligned}$$

(b) Using specified end slopes of 1 at $x = 1$ and $1/6$ at $x = 6$ (NOTE: There was a typo on the Web, the end slope at $x = 1$ is not zero!)

We need to set the values of y_1'' & y_5'' to values calculated from equation [2] from slide 78 in the notes so that y_1' & y_5' have the desired values. For the left endpoint, use $x = x_j$. For the right endpoint, use $x = x_{j+1}$.

$$\frac{dy}{dx} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{3A^2 - 1}{6}(x_{j+1} - x_j)y_j'' + \frac{3B^2 - 1}{6}(x_{j+1} - x_j)y_{j+1}''$$

$$\text{where } A = \frac{x_{j+1} - x}{x_{j+1} - x_j} \text{ and } B = 1 - A$$

$$\begin{aligned} \text{At } x = 1, \frac{dy}{dx} = 1: \quad & 1 = \frac{\ln 2 - \ln 1}{2 - 1} - \frac{3(1)^2 - 1}{6}(2 - 1)y_1'' + \frac{3(0)^2 - 1}{6}(2 - 1)y_2'' \\ & 1 = \ln 2 - \frac{1}{3}y_1'' - \frac{1}{6}y_2'' \end{aligned}$$

$$\begin{aligned} \text{At } x = 6, \frac{dy}{dx} = \frac{1}{6}: \quad & \frac{1}{6} = \frac{\ln 6 - \ln 4}{6 - 4} - \frac{3(0)^2 - 1}{6}(6 - 4)y_4'' + \frac{3(1)^2 - 1}{6}(6 - 4)y_5'' \\ & \frac{1}{6} = \frac{1}{2}\ln \frac{3}{2} + \frac{1}{3}y_4'' + \frac{2}{3}y_5'' \end{aligned}$$

With these conditions, "System 1" simplifies to:

$$\begin{aligned} \frac{1}{3}y_1'' + \frac{1}{6}y_2'' &= \ln 2 - 1 \\ \frac{1}{6}y_1'' + \frac{2}{3}y_2'' + \frac{1}{6}y_3'' &= \ln \frac{3}{4} & y_1'' &= -0.8216 \\ \frac{1}{6}y_2'' + \frac{2}{3}y_3'' + \frac{1}{6}y_4'' &= \ln \frac{8}{9} & y_2'' &= -0.1979 \\ \frac{1}{6}y_3'' + y_4'' + \frac{1}{3}y_5'' &= \ln \frac{3\sqrt{6}}{8} & y_3'' &= -0.1128 \\ & & y_4'' &= -0.05775 \\ & & y_5'' &= -0.02522 \\ \frac{1}{3}y_4'' + \frac{2}{3}y_5'' &= \frac{1}{6} - \ln \sqrt{\frac{3}{2}} \end{aligned}$$

The spline equation, which applied for the first subinterval $1 < x < 2$, is then given by:

$$\begin{aligned} y &= A_1y_1 + B_1y_2 + C_1y_1'' + D_1y_2'' \\ &= (2 - x)(\ln 1) + (x - 1)(\ln 2) + \frac{1}{6}[(2 - x)^3 - (2 - x)](-0.8216) + \frac{1}{6}[(x - 1)^3 - (x - 1)](-0.1979) \\ &= -0.03298(x - 1)^3 + 0.1369(x - 2)^3 + 0.7261(x - 1) - 0.1369(x - 2) \end{aligned} \quad S1$$

The spline equation, which applied for the second subinterval $2 < x < 3$, is then given by:

$$\begin{aligned}
 y &= A_2y_2 + B_2y_3 + C_2y_2'' + D_2y_3'' \\
 &= (3-x)(\ln 2) + (x-2)(\ln 3) + \frac{1}{6}[(3-x)^3 - (3-x)](-0.1979) + \frac{1}{6}[(x-2)^3 - (x-2)](-0.1128) \\
 &= -0.0188(x-2)^3 + 0.03298(x-3)^3 + 1.1174(x-2) - 0.7261(x-3) \quad \text{S2}
 \end{aligned}$$

The spline equation, which applied for the third subinterval $3 < x < 4$, is then given by:

$$\begin{aligned}
 y &= A_3y_3 + B_3y_4 + C_3y_3'' + D_3y_4'' \\
 &= (4-x)(\ln 3) + (x-3)(\ln 4) + \frac{1}{6}[(4-x)^3 - (4-x)](-0.1128) + \frac{1}{6}[(x-3)^3 - (x-3)](-0.05775) \\
 &= -0.009625(x-3)^3 + 0.0188(x-4)^3 + 1.3959(x-3) - 1.1174(x-4) \quad \text{S3}
 \end{aligned}$$

The spline equation, which applied for the fourth subinterval $4 < x < 6$, is then given by:

$$\begin{aligned}
 y &= A_4y_4 + B_4y_5 + C_4y_4'' + D_4y_5'' \\
 &= (3 - \frac{1}{2}x)(\ln 4) + (\frac{1}{2}x - 2)(\ln 6) + \frac{2}{3}[(3 - \frac{1}{2}x)^3 - (3 - \frac{1}{2}x)](-0.05775) \\
 &\quad + \frac{2}{3}[(\frac{1}{2}x - 2)^3 - (\frac{1}{2}x - 2)](-0.02522) \\
 &= -0.002102(x-4)^3 + 0.004813(x-6)^3 + 0.9043(x-4) - 0.7124(x-6) \quad \text{S4}
 \end{aligned}$$

(c) Using not-a-knot conditions...

We wish to enforce the continuity of the third derivative at the first and last internal knots (i.e., $j = 2$ and $j = 4$). To begin, we take the derivative of the second derivative (found on slide 79):

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(Ay_j'' + By_{j+1}'') = \frac{y_{j+1}' - y_j''}{x_{j+1} - x_j}$$

Continuity at these internal knots mean:

$$\text{At } j = 2: \quad \frac{y_2' - y_1''}{x_2 - x_1} = \frac{y_3' - y_2''}{x_3 - x_2} \Rightarrow \frac{y_2' - y_1''}{2-1} = \frac{y_3' - y_2''}{3-2} \Rightarrow y_1'' - 2y_2'' + y_3'' = 0$$

$$\text{At } j = 4: \quad \frac{y_4' - y_3''}{x_4 - x_3} = \frac{y_5' - y_4''}{x_5 - x_4} \Rightarrow \frac{y_4' - y_3''}{4-3} = \frac{y_5' - y_4''}{6-4} \Rightarrow y_3'' - \frac{3}{2}y_4'' + \frac{1}{2}y_5'' = 0$$

With these conditions, "System 1" simplifies to:

$$\begin{aligned}
 y_1'' - 2y_2'' + y_3'' &= 0 \\
 \frac{1}{6}y_1'' + \frac{2}{3}y_2'' + \frac{1}{6}y_3'' &= \ln \frac{3}{4} & y_1'' &= -0.4868 \\
 \frac{1}{6}y_2'' + \frac{2}{3}y_3'' + \frac{1}{6}y_4'' &= \ln \frac{8}{9} & y_2'' &= -0.2877 \\
 \frac{1}{6}y_3'' + y_4'' + \frac{1}{3}y_5'' &= \ln \frac{3\sqrt{6}}{8} & y_3'' &= -0.08860 \\
 y_3'' - \frac{3}{2}y_4'' + \frac{1}{2}y_5'' &= 0 & y_4'' &= -0.06462 \\
 & & y_5'' &= -0.01668
 \end{aligned}
 \Rightarrow$$

The spline equation, which applied for the first subinterval $1 < x < 2$, is then given by:

$$\begin{aligned}
 y &= A_1y_1 + B_1y_2 + C_1y_1'' + D_1y_2'' \\
 &= (2-x)(\ln 1) + (x-1)(\ln 2) + \frac{1}{6}[(2-x)^3 - (2-x)](-0.4868) + \frac{1}{6}[(x-1)^3 - (x-1)](-0.2877) \\
 &= -0.04795(x-1)^3 + 0.08113(x-2)^3 + 0.7411(x-1) - 0.08113(x-2) \quad S1
 \end{aligned}$$

The spline equation, which applied for the second subinterval $2 < x < 3$, is then given by:

$$\begin{aligned}
 y &= A_2y_2 + B_2y_3 + C_2y_2'' + D_2y_3'' \\
 &= (3-x)(\ln 2) + (x-2)(\ln 3) + \frac{1}{6}[(3-x)^3 - (3-x)](-0.2877) + \frac{1}{6}[(x-2)^3 - (x-2)](-0.0886) \\
 &= -0.01477(x-2)^3 + 0.04795(x-3)^3 + 1.1134(x-2) - 0.7411(x-3) \quad S2
 \end{aligned}$$

The spline equation, which applied for the third subinterval $3 < x < 4$, is then given by:

$$\begin{aligned}
 y &= A_3y_3 + B_3y_4 + C_3y_3'' + D_3y_4'' \\
 &= (4-x)(\ln 3) + (x-3)(\ln 4) + \frac{1}{6}[(4-x)^3 - (4-x)](-0.0886) + \frac{1}{6}[(x-3)^3 - (x-3)](-0.06462) \\
 &= -0.01077(x-3)^3 + 0.01477(x-4)^3 + 1.3971(x-3) - 1.1134(x-4) \quad S3
 \end{aligned}$$

The spline equation, which applied for the fourth subinterval $4 < x < 6$, is then given by:

$$\begin{aligned}
 y &= A_4y_4 + B_4y_5 + C_4y_4'' + D_4y_5'' \\
 &= (3 - \frac{1}{2}x)(\ln 4) + (\frac{1}{2}x - 2)(\ln 6) + \frac{2}{3}[(3 - \frac{1}{2}x)^3 - (3 - \frac{1}{2}x)](-0.06462) \\
 &\quad + \frac{2}{3}[(\frac{1}{2}x - 2)^3 - (\frac{1}{2}x - 2)](-0.01668) \\
 &= -0.00139(x-4)^3 + 0.005385(x-6)^3 + 0.9014(x-4) - 0.7147(x-6) \quad S4
 \end{aligned}$$

Please note that with this last case, since we are using the not-a-knot conditions, S1 and S2 are in fact the same cubic function, as is the case with S3 and S4. Therefore, if you are confident in your results, you actually only need to find two spline equations for this question when using the not-a-knot conditions! ☺

In each case, the four equations S1, S2, S3 and S4 constitute the natural cubic spline equations for y over the interval $1 < x < 6$.

The following are the plots of these splines in Matlab. The built-in spline function in Matlab is capable of finding splines using the fixed endslopes condition and the not-a-knot condition.

